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Best Approximation Using a Peak Norm

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A family of norms $\|g\|^{(\alpha)}$, $0<\alpha<1$, which combine features of both the uniform and the L^1 norms is defined. Best approximation of a continuous function from an n-dimensional subspace is characterized and (in case of a T-subspace) a uniqueness theorem is proven. The family, as well as the best approximation, is continuous in α. In particular, when α tends to zero or one, we get the uniform or the L^1 case, respectively.

1. INTRODUCTION

The uniform (L^∞, Chebyshev) norm $\max_{0\leqslant x\leqslant 1}|g(x)|$ measures the largest deviation of the continuous function g from 0, whereas the L^1 norm $\int_0^1 |g(x)|\,dx$ measures the average deviation. We use a class of norms, denoted by $\|g\|^{(\alpha)}$ where $0<\alpha<1$, which combine features of these two classical norms. Our $\|g\|^{(\alpha)}$, defined in Section 2, measures the average of the largest function values $|g(x)|$. As $\alpha\to 1^-$, $\|g\|^{(\alpha)}$ converges to the L^1 norm of g; as $\alpha\to 0^+$, $\|g\|^{(\alpha)}$ converges to the uniform norm of g. Corresponding results hold for best approximations to a given continuous function f.

Our main result is an L^1-type characterization theorem for best approximation. Interestingly, we obtain uniqueness of the best approximation from a Chebyshev system by an argument which uses both L^1 and uniform norm techniques.

Our work is somewhat in the spirit of [5]. There L^q-type gauges were introduced and a theory developed for $q=1$ reminiscent of best uniform approximation.

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2. Preliminaries and an Example

For each α with $0<\alpha\leqslant 1$ we define the *peak norm* or *α-norm* $\|\bullet\|^{(\alpha)}$ on the space $C[0,1]$ of real continuous functions g on $[0,1]$ by

$$\|g\|^{(\alpha)}=\frac{1}{\alpha}\sup_{m(A)=\alpha}\int_A |g|,$$

where the supremum is taken over all subsets A of $[0,1]$ with (Lebesgue) measure $m(A)=\alpha$. It is easy to verify that $\|\bullet\|^{(\alpha)}$ is in fact a norm on $C[0,1]$. Of course when $\alpha=1$, $\|g\|^{(\alpha)}$ is equal to the L^1 norm of g. For each $0<\alpha<1$ our $\|\bullet\|^{(\alpha)}$ is topologically equivalent to the L^1 norm on $C[0,1]$, since $\alpha\|g\|^{(\alpha)}\leqslant\int_0^1|g(x)|\,dx\leqslant\|g\|^{(\alpha)}$. Also $\|\bullet\|^{(\alpha)}$ is a monotone norm; i.e., if $|g(x)|\leqslant|f(x)|$, $0\leqslant x\leqslant 1$, then $\|g\|^{(\alpha)}\leqslant\|f\|^{(\alpha)}$. Finally note that $\|\bullet\|^{(\alpha)}$ is not strictly convex; this is easily shown by an example.

More generally, for $1\leqslant q<\infty$ we could define

$$\|g\|_q^{(\alpha)}=\left[\frac{1}{\alpha}\sup_{m(A)=\alpha}\int_A |g|^q\right]^{1/q}$$

and obtain results similar to the case $q=1$ studied in this paper.

Our first result concerns existence and structure of sets A' for which $m(A')=\alpha$ and $(1/\alpha)\int_{A'}|g|=\|g\|^{(\alpha)}$. Intuitively, A' is a set of x-values (of measure α) corresponding to the largest $|g(x)|$ values. Throughout this paper we will denote the set difference of two sets by $A\backslash B=A\cap(B^C)$ and the symmetric difference by $A\triangle B=(A\backslash B)\cup(B\backslash A)$.

We use the following notations: Let g be a continuous function on $[0,1]$. For h real, set

$$A_h(g)=\{x\in[0,1]:|g(x)|\geqslant h\}$$

$$h_0(g,\alpha)=\inf\{h:m(A_h(g))\leqslant\alpha\}$$

and

$$A_{h0}^+(g)=\{x\in[0,1]:|g(x)|>h_0\}.$$

Lemma 1. *Let $0<\alpha<1$ and g be a continuous function on $[0,1]$, $A_h=A_h(g)$, $h_0=h_0(g,\alpha)$, and $A_{h0}^+=A_{h0}^+(g)$. Then*

(1) $m(A_{h0}^+)\leqslant\alpha\leqslant m(A_{h_0})$.

(2) *There exists a set $A'\subseteq[0,1]$ with $m(A')=\alpha$ and $(1/\alpha)\int_{A'}|g|=(1/\alpha)\sup_{m(A)=\alpha}\int_A|g|=\|g\|^{(\alpha)}$. In fact any set A' with $A_{h0}^+\subseteq A'\subseteq A_{h_0}$ and $m(A')=\alpha$ is such that $(1/\alpha)\int_{A'}|g|=\|g\|^{(\alpha)}$.*

(3) *Conversely, if* $A' \subseteq [0, 1]$ *and* $m(A') = \alpha$ *and if* $(1/\alpha) \int_{A'} |g| = \|g\|^{(\alpha)}$ *then (except possibly for a set of measure* 0) $A^+_{h_0} \subseteq A' \subseteq A_{h_0}$.

The proof of Lemma 1 is straightforward and will be omitted.

We define a *norming set for* g (using $\|\bullet\|^{(\alpha)}$) to be any set A' with $m(A') = \alpha$ and $A^+_{h_0} \subseteq A' \subseteq A_{h_0}$ (where h_0, A_{h_0}, $A^+_{h_0}$ are defined in the statement of Lemma 1). If $m(A^+_{h_0}) = \alpha$ or if $m(A_{h_0}) = \alpha$ then g has a unique (up to a set of measure 0) norming set. If $m(A^+_{h_0}) < \alpha < m(A_{h_0})$ then g does not have a unique norming set, since any set of the form $A' = A^+_{h_0} \cup E$, where $E \subseteq \{x \in [0, 1] : |g(x)| = h_0\}$ and $m(E) = \alpha - m(A^+_{h_0})$, is a norming set for g. Possible nonuniqueness of norming sets is a complicating feature in the analysis below. Finally, note that for each norming set A' for g,

$$h_0 = \inf_{x \in A'} |g(x)|.$$

This follows from the continuity of g.

For the linearly independent continuous functions $u_1, ..., u_n$ on $[0, 1]$, set $U = \text{span}\{u_1, ..., u_n\}$ = the n-dimensional subspace spanned by $u_1, ..., u_n$. Then u^* in U is a *best peak norm* (or *best* α*-norm*) approximation to f in $C[0, 1]$ from U if $\|f - u^*\|^{(\alpha)} \leqslant \|f - u\|^{(\alpha)}$ for all u in U. Existence of a best peak norm approximation to f from the finite dimensional subspace U is guaranteed by a standard existence theorem, cf. [2, p. 20].

We next present an example.

Example. Let $0 < \alpha < 1$. We seek a best α-norm approximation to $f(x) = (x - 1/2)^2$, $0 \leqslant x \leqslant 1$, using $c_1 + c_2 x$. Motivated by symmetry we try $c_2^* = 0$ and

$$c_1^* = \left[f\left(\frac{\alpha}{4}\right) + f\left(\frac{1}{2} - \frac{\alpha}{4}\right)\right]\Big/2 = \left[\left(\frac{\alpha}{4} - \frac{1}{2}\right)^2 + \left(-\frac{\alpha}{4}\right)^2\right]\Big/2$$
$$= [(\alpha - 2)^2 + \alpha^2]/32.$$

Notice $A' = [0, \alpha/4] \cup [1/2 - \alpha/4,\ 1/2 + \alpha/4] \cup [1 - \alpha/4, 1]$ is a norming set for $f(x) - (c_1^* + c_2^* x)$.

It follows from Theorem 1 in Section 3 below that $u^*(x) = c_1^* + c_2^* x$ is in fact a best α-norm approximation.

Notice $\lim_{\alpha \to 0^+} [f(x) - (c_1^* + c_2^* x)] = (x - 1/2)^2 - 1/8$, a multiple of the Chebyshev polynomial of the first kind $T_2(t) = 2t^2 - 1$, $-1 \leqslant t \leqslant 1$, transformed to $0 \leqslant x \leqslant 1$ by the change of variable $t = -1 + 2x$. Also $\lim_{\alpha \to 1^-} [f(x) - (c_1^* + c_2^* x)] = (x - 1/2)^2 - 1/16$, a multiple of the Chebyshev polynomial of the second kind $U_2(t) = 4t^2 - 1$, $-1 \leqslant t \leqslant 1$, transformed to $0 \leqslant x \leqslant 1$. These results are instances of Theorem 4 below.

3. Main Results

The next theorem is our main result. When $\alpha = 1$ the criterion (3.1) reduces to that of a well-known characterization theorem for L^1 approximation on [0, 1], cf. [4, p. 104].

THEOREM 1 (L^1-Type Characterization Theorem). *Let* $0 < \alpha < 1$, f, $u_1, ..., u_n$ *continuous on* $[0, 1]$, *and* $U = \text{span}\{u_1, ..., u_n\}$. *Let* $u^* \in U$ *and set* $Z = \{x \in [0, 1] : f(x) - u^*(x) = 0\}$, *the zero set of* $f - u^*$. *Then* u^* *is a best peak norm approximation to* f *from* U *if and only if for each* u *in* U *there exists a norming set* $A(u)$ *for* $f - u^*$ *such that*

$$\int_{A(u)} u \operatorname{sgn}(f - u^*) \leqslant \int_{Z \cap A(u)} |u|. \tag{3.1}$$

Proof. The proof is presented in Section 5.

Remarks. (1) It can be shown that Theorem 1 remains valid if absolute value signs are placed around the integral on the left-hand side of (3.1). Hence if $h_0 > 0$ then $Z \cap A(u) = \phi$ and (3.1) becomes

$$\int_{A(u)} u \operatorname{sgn}(f - u^*) = 0.$$

(2) If u^* is a best L^1 approximation to f on a norming set A for $f - u^*$ (i.e., if $\int_A |f - u^*| \leqslant \int_A |f - u|$ for all u in U) then u^* is a best peak norm approximation to f. If $f - u^*$ has a unique (up to a set of measure 0) norming set A, then the converse is true: if u^* is a best peak norm approximation to f then u^* is a best L^1 approximation to f on A. These facts follow from Theorem 1 and from a characterization theorem for L^1 approximation on the set A.

(3) (a) If u^* is a best α-norm approximation to f with $h_0 = \inf_{x \in A} |f(x) - u^*(x)| = 0$ (A is a norming set) then u^* is also a best β-approximation to f for each β with $\alpha < \beta \leqslant 1$. This is a direct consequence of Theorem 1 since now $f(x) - u^*(x) = 0$ for all x in $[0, 1] \backslash A$. This can also be shown without using Theorem 1 as follows. For any u in U,

$$\|f - u^*\|^{(\beta)} = \frac{\alpha}{\beta} \|f - u^*\|^{(\alpha)} \leqslant \frac{\alpha}{\beta} \|f - u\|^{(\alpha)} \leqslant \|f - u\|^{(\beta)}.$$

(b) If u^* is a best L^1 approximation to f on [0, 1] and if $m\{x : |f(x) - u^*(x)| > 0\} \leqslant \alpha < 1$ then it does *not* follow that u^* is a best α-norm approximation to f. This is easily seen by example.

Our next theorem gives intuitively appealing "uniform approximation type" properties of a best peak norm approximation. First, the set $\{u_1, ..., u_n\}$ of continuous functions on $[0, 1]$ is a *Chebyshev system* on $[0, 1]$ if each linear combination $c_1 u_1 + \cdots + c_n u_n$ has fewer than n zeros in $[0, 1]$ unless $c_1 = 0, ..., c_n = 0$.

THEOREM 2. *Let f be continuous on $[0, 1]$ and $\{u_1, ..., u_n\}$ a Chebyshev system of continuous functions on $[0, 1]$. Let $0 < \alpha < 1$ and let u^* be a best α-norm approximation to f from $U = \operatorname{span}\{u_1, ..., u_n\}$. Set*

$$A_h = A_h(f - u^*)$$
$$h_0 = \inf_{x \in A} |f(x) - u^*(u)|,$$

where A is any norming set for $f - u^*$. If $h_0 > 0$ then there are closed sets $A^{(1)}, ..., A^{(m)}$ with $m \geqslant n+1$ such that:

(1) $A_{h_0} = \bigcup_{i=1}^{m} A^{(i)}$.

(2) $A^{(1)} < A^{(2)} < \cdots < A^{(m)}$ *and, in fact, there exists $d > 0$ such that* $\min A^{(i+1)} - \max A^{(i)} \geqslant d$, $i = 1, ..., m-1$.

(3) $\operatorname{sgn} A^{(i+1)} = -\operatorname{sgn} A^{(i)}$, $i = 1, ..., m-1$, *where*

$$\operatorname{sgn} A^{(i)} = \begin{cases} +1 & \text{if } f(x) - u^*(x) \geqslant h_0 \text{ for all } x \text{ in } A^{(i)} \\ -1 & \text{if } f(x) - u^*(x) \leqslant -h_0 \text{ for all } x \text{ in } A^{(i)}. \end{cases}$$

(4) *There exists a subsequence $A^{(i_1)}, A^{(i_2)}, ..., A^{(i_{m'})}$ of $A^{(1)}, ..., A^{(m)}$ with $m' \geqslant n+1$, $\operatorname{sgn} A^{(i_{j+1})} = -\operatorname{sgn} A^{(i_j)}$, $j = 1, ..., m'-1$, and $m(A^{(i_j)}) > 0$, $j = 1, ..., m'$.*

(5) *Set $t_i = \min A^{(i)}$, $i = 2, ..., m$, and $s_i = \max A^{(i)}$, $i = 1, ..., m-1$. Then*

(a) $|f(t_i) - u^*(t_i)| = h_0$, $i = 2, ..., m$. $|f(s_i) - u^*(s_i)| = h_0$, $i = 1, ..., m-1$.

(b) *u^* is the unique best uniform approximation on the finite point set $\{s_1, t_2, s_2, ..., t_{m-1}, s_{m-1}, t_m\}$ and also on any finite point set of the form $(s_1, r_2, ..., r_{m-1}, t_m\}$ where $r_i \in \{t_i, s_i\}$, $i = 2, ..., m-1$.*

Proof. By the uniform continuity of $f - u^*$ on $[0, 1]$, there exists $d > 0$ such that $|(f - u^*)(x) - (f - u^*)(y)| < 2h_0$ if $|x - y| \leqslant d$. Partition $[0, 1]$ into a finite number of subintervals I of length $\leqslant d$. Label I as a $+$subinterval if $f(x) - u^*(x) \geqslant h_0$ for some x in I, as a $-$subinterval if $f(x) - u^*(x) \leqslant -h_0$ for some x in I. (I may be neither $+$ nor $-$ but it cannot be both $+$ and $-$.) Starting at the left end of $[0, 1]$, form $A^{(1)}$ by intersecting A_{h_0} with successive subintervals I; stop when a subinterval of opposite sign is encountered. Then from $A^{(2)}$ using subintervals of opposite sign from $A^{(1)}$. Continue until all subintervals have been used. Then each $A^{(i)}$ is closed (since A_{h_0} is closed) and (1), (2), (3) are clear, except for $m \geqslant n+1$. We prove this by contradiction; assume $m \leqslant n$. If $m=1$, then $\operatorname{sgn}(f-u^*)$ does not change on A_{h_0}. There exists u in U with $u(x)>0$ for all x in $[0, 1]$ (because $\{u_1, ..., u_n\}$ is a Chebyshev system). Using either u or $-u$ we obtain a contradiction from $A(u) \subseteq A_{h_0}$ and Theorem 1 ($Z \cap A(u) = \phi$ there since $h_0 > 0$). If $2 \leqslant m \leqslant n$, let $x_1, ..., x_{m-1}$ be points satisfying

$$A^{(i)} < x_i < A^{(i+1)}, \qquad i = 1, ..., m-1.$$

Then there exists u in U which changes sign precisely at $x_1, ..., x_{m-1}$. Again using either u or $-u$ we obtain a contradiction from Theorem 1. Hence $m \geqslant n+1$. Part (4) is proved similarly.

Part (5(a)) follows from the closedness of $A^{(i)}$ and the continuity of $f-u^*$. Part (5(b)) is an immediate consequence of the alternation theorem and uniqueness theorem for best uniform approximation on a finite point set, cf. [2, p. 75; 6, Chap. 3]. ▮

In the example of Section 2, $A^{(1)} = [0, \alpha/4]$,

$$A^{(2)} = [1/2 - \alpha/4, 1/2 + \alpha/4],\ A^{(3)} = [1 - \alpha/4, 1].$$

The next theorem generalizes a classical uniqueness theorem of Jackson for L^1 approximation.

THEOREM 3 (Uniqueness). *Let* $0 < \alpha < 1$, f *continuous on* $[0, 1]$, $\{u_1, ..., u_n\}$ *a Chebyshev system of continuous functions on* $[0, 1]$, *and* $U = \operatorname{span}\{u_1, ..., u_n\}$. *Then the best* α*-norm approximation to* f *from* U *is unique.*

Proof. Assume p_1 and p_2 are two different best α-norm approximations to f from U and set $p_0 = (p_1 + p_2)/2$. Let A be a norming set for $f - p_0$. Then

$$\begin{aligned} \|f-p_0\|^{(\alpha)} &= \frac{1}{\alpha}\int_A |f-p_0| = \frac{1}{\alpha}\int_A |f-(p_1+p_2)/2| \\ &\leqslant \left[\frac{1}{\alpha}\int_A |f-p_1| + \frac{1}{\alpha}\int_A |f-p_2|\right]\Big/2 \\ &\leqslant [\|f-p_1\|^{(\alpha)} + \|f-p_2\|^{(\alpha)}]/2. \end{aligned} \tag{3.2}$$

Thus p_0 is also a best α-norm approximation to f, both $\leqslant$ are $=$, and A

is (up to a set of measure 0) a norming set for $f-p_1$ and for $f-p_2$. The fact that inequality in (3.2) is equality implies

$$|(f(x)-p_1(x))+(f(x)-p_2(x))| = |f(x)-p_1(x)| + |f(x)-p_2(x)| \tag{3.3}$$

almost everywhere on A. Now for $j=0, 1, 2$ define $A_{h^{(j)}} = A_{h^{(j)}}(f-p_j)$, $h_0^{(j)} = h_0(f-p_j, \alpha)$, $A^+_{h_0^{(j)}} = A^+_{h_0^{(j)}}(f-p_j)$.

Let A' be a subset of A of measure α on which (3.3) holds and which is a norming set for $f-p_1$ and for $f-p_2$. Then

$$\begin{aligned} h_0^{(0)} &= \inf_{x\in A'} |f(x)-p_0(x)| \\ &= \tfrac{1}{2} \inf_{x\in A'} [|f(x)-p_1(x)| + |f(x)-p_2(x)|] \\ &\geqslant \tfrac{1}{2} [\inf_{x\in A'} |f(x)-p_1(x)| + \inf_{x\in A'} |f(x)-p_2(x)|] \\ &= \tfrac{1}{2} [h_0^{(1)} + h_0^{(2)}]. \end{aligned} \tag{3.4}$$

If $h_0^{(0)}=0$, then $h_0^{(1)}=0=h_0^{(2)}$ also and so $p_1(x)=f(x)=p_2(x)$ for all x in $[0, 1]\backslash A'$. This is impossible since $\{u_1, ..., u_n\}$ is a Chebyshev system. Hence $h_0^{(0)}>0$. Let t_i, s_i be defined as in part (5) of Theorem 2, with p_0 replacing u^* there and $h_0^{(0)}$ replacing h_0 there. Since $A^+_{h_0^{(1)}} \subseteq A' \subseteq A^+_{h_0^{(0)}}$, every interval of the form $(s_1, s_1+\varepsilon)$ with $\varepsilon>0$ contains points y not in $A^+_{h_0^{(1)}}$. Let $y_1, y_2, ...$ be a sequence of such points with $\lim_{j\to\infty} y_j = s_1$. Then since $|f(y_j)-p_1(y_j)| = h_0^{(1)}$ we have

$$|f(s_1)-p_1(s_1)| = \lim_{j\to\infty} |f(y_j)-p_1(y_j)| \leqslant h_0^{(1)}. \tag{3.5}$$

Similarly, $|f(s_1)-p_2(s_1)| \leqslant h_0^{(2)}$.

From (3.4) one of $h_0^{(1)}$, $h_0^{(2)}$ is $\leqslant h_0^{(0)}$; say $h_0^{(1)} \leqslant h_0^{(0)}$. Then from (3.5), $|f(s_1)-p_1(s_1)| \leqslant h_0^{(1)} \leqslant h_0^{(0)}$. In a similar way to that in which (3.5) was obtained, we can get $|f(s_i)-p_1(s_i)| \leqslant h_0^{(1)} \leqslant h_0^{(0)}$, $i=2, ..., m-1$, and $|f(t_m)-p_1(t_m)| \leqslant h_0^{(1)} \leqslant h_0^{(0)}$. Hence using the uniqueness part of (5(b)) of Theorem 2 we see $p_1 = p_0$. Also $p_2 = 2p_0 - p_1 = p_1$. This contradiction completes the proof. ▌

4. Continuous Dependence on α

In this section we consider the dependence on α of a best α-norm approximation to f. We first state a lemma whose proof is straightforward and will be omitted.

LEMMA 2. *Let g be a continuous function on* $[0, 1]$.

(1) *If* $0<\beta<\alpha\leqslant 1$, *then* $\|g\|^{(\alpha)}\leqslant \|g\|^{(\beta)}\leqslant (\alpha/\beta)\, \|g\|^{(\alpha)}$.

(2) *If* $0<\alpha<1$, *then* $\lim_{\beta\to\alpha} \|g\|^{(\beta)}=\|g\|^{(\alpha)}$.

(3) (a) $\lim_{\beta\to 1^-} \|g\|^{(\beta)}=\int_0^1 |g(x)|\, dx$.

(b) $\lim_{\beta\to 0^+} \|g\|^{(\beta)}=\max_{0\leqslant x\leqslant 1} |g(x)|$.

THEOREM 4. *Let f be continuous on* $[0, 1]$, $\{u_1, ..., u_n\}$ *a Chebyshev system of continuous functions on* $[0, 1]$, *and* $U=\text{span}\{u_1, ..., u_n\}$. *Let* p_α (p_β) *be the unique best α-norm (β-norm) approximation to f from U.*

(1) *If* $0<\beta<\alpha\leqslant 1$ *then*

$$\|f-p_\alpha\|^{(\alpha)}\leqslant \|f-p_\beta\|^{(\alpha)}\leqslant \|f-p_\beta\|^{(\beta)}\leqslant \|f-p_\alpha\|^{(\beta)}\leqslant\frac{\alpha}{\beta}\, \|f-p_\alpha\|^{(\alpha)}.$$

(2) *If* $0<\alpha<1$, *then* $\lim_{\beta\to\alpha} p_\beta=p_\alpha$.

(3) (a) $\lim_{\beta\to 1^-} p_\beta=p_1$.

(b) $\lim_{\beta\to 0^+} p_\beta=p_0$, *the best uniform approximation to f on* $[0, 1]$ *from U.*

Proof. The first inequality of (1) follows from the definition of p_α, the second inequality from Lemma 2, the third inequality from the definition of p_β, and the fourth inequality from Lemma 2. Parts (2), (3) are immediate consequence of the following (special case of *a*) result from [3].

Let X be a normed linear space with norm $\|\bullet\|$, V a finite dimensional subspace of X, f in X, and $\|\bullet\|_k$, $k=1, 2, ...$, norms on X which satisfy $\lim_{k\to\infty} \|g\|_k=\|g\|$ for each g in X. If v^* is the unique best approximation to f from V using $\|\bullet\|$ and if v_k is a best approximation to f from V using $\|\bullet\|_k$ then $\lim_{k\to\infty} v_k=v^*$. ∎

5. PROOF OF THE CHARACTERIZATION THEOREM

In this section we present the proof of our main result. Our proof is patterned on the proof of the characterization theorem for L^1 approximation in [7, p. 67].

Proof of Theorem 1. ($\Leftarrow$) Let $\hat{u}\in U$ and let $A(\hat{u}-u^*)$ be a norming set for $f-u^*$ such that (3.1) holds with $u=\hat{u}-u^*$. Then

$$\begin{aligned}
\alpha \|f-u^*\|^{(\alpha)} &= \int_{A(\hat{u}-u^*)} |f-u^*| = \int_{A(\hat{u}-u^*)} (f-u^*)\operatorname{sgn}(f-u^*) \\
&= \int_{A(\hat{u}-u^*)} (f-\hat{u})\operatorname{sgn}(f-u^*) + \int_{A(\hat{u}-u^*)} (\hat{u}-u^*)\operatorname{sgn}(f-u^*) \\
&\leqslant \int_{A(\hat{u}-u^*)\setminus Z} (f-u^*)\operatorname{sgn}(f-u^*) + \int_{A(\hat{u}-u^*)\cap Z} |\hat{u}-u^*| \qquad \text{by (3.1)} \\
&\leqslant \int_{A(\hat{u}-u^*)\setminus Z} |f-\hat{u}| + \int_{A(\hat{u}-u^*)\cap Z} |\hat{u}-f| \qquad \text{since } u^*=f \text{ on } Z \\
&= \int_{A(\hat{u}-u^*)} |f-\hat{u}| \leqslant \alpha \|f-\hat{u}\|^{(\alpha)}.
\end{aligned}$$

Thus u^* is a best α-norm approximation to f from U.

($\Rightarrow$) We give a proof by contradiction. Let u^* in U be a best α-norm approximation to f from U and assume there exists u in U such that

$$\int_A u \operatorname{sgn}(f-u^*) - \int_{Z\cap A} |u| > 0 \tag{5.1}$$

for every norming set A for $f-u^*$. We can scale u so that

$$\max_{0\leqslant x\leqslant 1} |u(x)| = 1.$$

Set $A_{h0}^+ = \{x \in [0,1] : |f(x)-u^*(x)| > h_0\}$. Recall (from Section 2) $m(A_{h0}^+) \leqslant \alpha \leqslant m(A_{h0})$. The proof will be accomplished by four assertions.

1. *There exists* $a > 0$ *such that*

$$\int_A u \operatorname{sgn}(f-u^*) - \int_{Z\cap A} |u| \geqslant a \tag{5.2}$$

for every norming set A for $f-u^*$.

Proof of 1. If $m(A_{h0}^+) = \alpha$ or if $\alpha = m(A_{h0})$ then there is a unique norming set A (up to a set of measure 0) and so (5.2) follows from (5.1). Now consider $m(A_{h0}^+) < \alpha < m(A_{h0})$. Each norming set A can be written $A = A_{h0}^+ \cup E_A$ *where* E_A *is a subset of* $A_{h0} \setminus A_{h0}^+ = \{x \in [0,1] : |f(x)-u^*(x)| = h_0\}$ with $m(E_A) = \alpha - m(A_{h0}^+)$ (see Section 2).

Case 1. $h_0 > 0$. Then $Z \cap A = \phi$ and (5.2) becomes

$$\int_A u \operatorname{sgn}(f - u^*) \geqslant a \tag{5.3}$$

for every norming set A for $f - u^*$. To show this, define

$$N_t = \{x \in A_{h_0} \backslash A_{h_0}^+ : u(x) \operatorname{sgn}(f - u^*)(x) \leqslant t\}$$
$$t_0 = \sup\{t : m(N_t) \leqslant \alpha - m(A_{h_0}^+)\}$$
$$N_{\bar{t}_0} = \{x \in A_{h_0} \backslash A_{h_0}^+ : u(x) \operatorname{sgn}(f - u^*)(x) < t_0\}.$$

Then the infinum of $\int_A u \operatorname{sgn}(f - u^*)$ is attained on a norming set $A = A_{h_0}^+ \cup N_{\bar{t}_0} \cup N_A$ where N_A is any subset of $N_{t_0} \backslash N_{\bar{t}_0}$ of measure $m(N_A) = \alpha - m(A_{h_0}^+) - m(N_{\bar{t}_0})$. Since $u(x) \operatorname{sgn}(f - u^*)(x) = t_0$ on $N_{t_0} \backslash N_{\bar{t}_0}$, $\int_A u \operatorname{sgn}(f - u^*)$ is the same for all such N_A; this establishes (5.3).

Case 2. $h_0 = 0$. Here

$$\int_A u \operatorname{sgn}(f - u^*) - \int_{Z \cap A} |u| = \int_{A_0^+} u \operatorname{sgn}(f - u^*) - \int_{A \backslash A_0^+} |u|.$$

The first integral on the right hand side is independent of A. As an Case 1 we can show $\inf_A [-\int_{A \backslash A_0^+} |u|]$ is attained by a norming set A for $f - u^*$. This completes the proof of 1. ∎

2. *There exists an open set G of real numbers such that* $A_{h_0} \subseteq G$ *and an open set B such that* $Z \subseteq B \subseteq G$, $m(B \backslash Z) < a/4$ *and*

$$\int_{\hat{A} \backslash B} u \operatorname{sgn}(f - u^*) - \int_{\hat{A} \cap B} |u| > \frac{a}{2} \tag{5.4}$$

whenever $A_{h_0}^+ \subseteq \hat{A} \subseteq G \cap [0, 1]$ *and* $m(\hat{A}) = \alpha$.

Proof of 2. If $h_0 > 0$ then $Z = \phi$ and we can take $B = \phi$. Then 2 follows from 1. If $h_0 = 0$ then $A_0 = [0, 1]$ and 2 follows from 1 and the two inequalities

$$\int_{\hat{A} \cap B} |u| < \int_{\hat{A} \cap Z} |u| + \frac{a}{4} \qquad \text{and} \qquad \left| \int_{\hat{A} \cap B} u \operatorname{sgn}(f - u^*) \right| < \frac{a}{4}$$

($\int_{\hat{A} \backslash B} u \operatorname{sgn}(f - u^*) - \int_{\hat{A} \cap B} |u| \geqslant \int_{\hat{A}} u \operatorname{sgn}(f - u^*) - \int_{\hat{A} \cap B} u \operatorname{sgn}(f - u^*) - \int_{\hat{A} \cap Z} |u| - a/4 \geqslant a - a/4 - a/4 = a/2$). ∎

3. *There exists* $\delta_0 > 0$ *such that if* $|\delta| \leqslant \delta_0$ *and if* $u_0 = u^* + \delta u$ *and if* $\tilde{A}$

is a norming set for $f-u_0$, *then* $\tilde{A}\subseteq G$ *and there exists* $\hat{A}$ *such that* $A^+_{h_0}\subseteq \hat{A}\subseteq (G\cap[0,1])$, $m(\hat{A})=\alpha$, $m(\tilde{A}\triangle\hat{A})<a/8$,

$$\left|\int_{\tilde{A}\setminus B} u\,\mathrm{sgn}(f-u^*)-\int_{\hat{A}\setminus B} u\,\mathrm{sgn}(f-u^*)\right|\leqslant m(\tilde{A}\triangle\hat{A})<\frac{a}{8} \tag{5.5}$$

and

$$\left|\int_{\tilde{A}\cap B}|u|-\int_{\hat{A}\cap B}|u|\right|\leqslant m(\tilde{A}\triangle\hat{A})<\frac{a}{8}. \tag{5.6}$$

The proof of 3 is straightforward and will be omitted.

4. *There exist* $\delta_1>0$ *such that for* $0<\delta<\delta_1$ *and* $u_0=u^*+\delta u$ *we have* $\|f-u_0\|^{(\alpha)}<\|f-u^*\|^{(\alpha)}$. (*This contradicts the assumption that* u^* *is a best* α-*norm approximation to* f *from* U *and completes the proof of Theorem* 1.)

Proof of 4. Let $A''=[G\setminus B]\cap[0,1]$. Then there exists $M>0$ such that $|f(x)-u^*(x)|\geqslant M$ for all x in A''. (If $h_0=0$ then $G\supseteq A_{h_0}=[0,1]$ and $A''=G\cap B^c\cap[0,1]=B^c\cap[0,1]$ is closed. Since $|f(x)-u^*(x)|>0$ on A'', then $\inf_{x\in A''}|f(x)-u^*(x)|>0$. If $h_0>0$ then $|f(x)-u^*(x)|\geqslant h_0/2$ for x in G). Then for $0<\delta<M$ we have, for $u_0=u^*+\delta u$, that $\mathrm{sgn}(f-u^*)=\mathrm{sgn}(f-u_0)$ on A''.

Let $\delta_1=\min\{\delta_0, M\}$, let $0<\delta<\delta_1$, let $\tilde{A}$ be a norming set for $f-u_0$, and let $\hat{A}$ be given by 2. Then

$$\begin{aligned}
\alpha\,\|f-u_0\|^{(\alpha)}&=\int_{\tilde{A}}|f-u_0|\\
&=\int_{\tilde{A}\cap B}|f-u_0|+\int_{\tilde{A}\setminus B}|f-u_0|\\
&=\int_{\tilde{A}\cap B}|f-u_0|+\int_{\tilde{A}\setminus B}(f-u_0)\,\mathrm{sgn}(f-u_0)\\
&=\int_{\tilde{A}\cap B}|f-u_0|+\int_{\tilde{A}\setminus B}(f-u_0)\,\mathrm{sgn}(f-u^*)\\
&=\int_{\tilde{A}\cap B}|f-u_0|+\int_{\tilde{A}\setminus B}(f-u^*)\,\mathrm{sgn}(f-u^*)\\
&\quad-\delta\int_{\tilde{A}\setminus B}u\,\mathrm{sgn}(f-u^*)
\end{aligned}$$

$$< \int_{\tilde{A} \cap B} |f - u_0| + \int_{\tilde{A} \setminus B} |f - u^*|$$

$$- \delta \left[\int_{\hat{A} \setminus B} u \operatorname{sgn}(f - u^*) - \frac{a}{8} \right] \qquad \text{by (5.5)},$$

$$< \int_{\tilde{A} \cap B} |f - u_0| + \int_{\tilde{A} \setminus B} |f - u^*|$$

$$- \delta \left[\int_{\hat{A} \cap B} |u| + \frac{a}{2} - \frac{a}{8} \right] \qquad \text{by (5.4)},$$

$$< \int_{\tilde{A} \cap B} |f - u_0| + \int_{\tilde{A} \cap B} |f - u^*|$$

$$- \delta \left[\int_{\tilde{A} \cap B} |u| - \frac{a}{8} + \frac{a}{2} - \frac{a}{8} \right] \qquad \text{by (5.6)},$$

$$< \int_{\tilde{A} \cap B} |f - u^*| + \int_{\tilde{A} \cap B} [|f - u_0| - |f - u^*| - \delta |u|] - \delta \frac{a}{4}$$

$$< \alpha \, \|f - u^*\|^{(\alpha)} \qquad \text{since the second integrand is} \leqslant 0$$

(since $\delta |u| = |-\delta u| = |(f - u_0) - (f - u^*)| \geqslant |f - u_0| - |f - u^*|$). This completes the proof of 4. ▮

Remark. In his report the referee commented that the α-norm had been discussed in [1], defined by $\|g\|^{(\alpha)} = (1/\alpha) \int_0^\alpha |g^*(t)| \, dt$ where g^* is a decreasing rearrangement of g. It is shown there (p. 109) that for every $g \in L^1[0, 1]$,

$$\|g\|^{(\alpha)} = \inf \left\{ \frac{1}{\alpha} \|g_1\|_1 + \|g_2\|_\infty : g_1 \in L^1, \, g_2 \in L^\infty, \, g_1 + g_2 = g \right\}.$$

The dual space is $L^\infty[0, 1]$ with the norm $\|\phi\|_{(\alpha)} = \max\{ \|\phi\|_1, \alpha \|\phi\|_\infty \}$ [1, p. 32]. Our Theorem 1 can be proved using a classic result on best approximation [8, p. 18]. The proof, however, is not short.

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